Learning, Confidence, and Business Cycles
Online Appendix

A Recursive competitive equilibrium

We define the recursive competitive equilibrium for the baseline model described in Section 2.. We collect exogenous aggregate state variables (such as economy-wide TFP) in a vector X with a cumulative transition function $F(X'|X)$. The endogenous aggregate state is the distribution of firm-level variables. A firm's type is identified by the posterior mean estimate of productivity $\tilde{z}_l$ and the posterior variance Σ_l. The worst-case TFP is not included because it is implied by the posterior mean and variance. We denote the cross-sectional distribution of firms' type by ξ_1 and ξ_2. ξ_1 is a stage 1 distribution over $(\tilde{z}_l, \Sigma_l)$ and ξ_2 is a stage 2 distribution over $(\tilde{z}_l', \Sigma_l')$. ξ_1', in turn, is a distribution over $(\tilde{z}_l', \Sigma_l')$ at stage 1 in the next period.[1]

First, consider the household's problem. The household's wealth can be summarized by a portfolio $\overrightarrow{\theta_l}$ which consists of share θ_l for each firm, capital stock K and the riskless bond holdings B. We use V_1^h and V_2^h to denote the household's value function at stage 1 and stage 2, respectively. We use m to summarize the income available to the household at stage 2. The household's problem at stage 1 is

$$\begin{aligned} V_1^h(\overrightarrow{\theta_l}, K, B; \xi_1, X) = \max_H & \left\{ -\frac{H^{1+\phi}}{1+\phi} + E^*[V_2^h(\hat{m}; \hat{\xi}_2, X)] \right\} \\ \text{s.t.} \quad & \hat{m} = WH + r^K K + RB + \int (\hat{D}_l + \hat{P}_l)\theta_l dl \end{aligned} \tag{1}$$

where we momentarily use the *hat* symbol to indicate random variables that will be resolved at stage 2. The household's problem at stage 2 is

$$\begin{aligned} V_2^h(m; \xi_2, X) = & \max_{C, \overrightarrow{\theta_l}', K', B'} \left\{ \ln C + \beta \int V_1^h(\overrightarrow{\theta_l}', K', B'; \xi_1', X') dF(X'|X) \right\} \\ \text{s.t.} \quad & C + K' - (1-\delta)K + B' + \int P_l \theta_l' dl \leq m \\ & \xi_1' = \Gamma(\xi_2, X) \end{aligned} \tag{2}$$

In problem (1), households choose labor supply based on the worst-case stage 2 value (recall that we use E^* to denote worst-case conditional expectations). The problem (2), in turn, describes the household's consumption and asset allocation problem given the realization of income and aggregate states. In particular, they take as given the law of motion of the next period's distribution $\xi_1' = \Gamma(\xi_2, X)$, which in equilibrium is consistent with the firm's policy function. Importantly, in contrast to the stage 2 problem, a law of motion that describes the evolution of ξ_2 from (ξ_1, X) is absent in the stage 1 problem. Indeed, if there is no ambiguity

[1] See also Senga (2018) for a recursive representation of an imperfect information heterogeneous-firm model with time-varying uncertainty.

in the model, agents take as given the law of motion $\xi_2 = \Upsilon(\xi_1, X)$, which in equilibrium is consistent with the firm's policy function and the true data generating process of the firm-level profitability. Since agents are ambiguous about each firm's profitability process, they cannot settle on a single law of motion about the distribution of firms. Finally, the continuation value at stage 2 is governed by the transition density of aggregate exogenous states X.

Next, consider the firms' problem. We use v_1^f and v_2^f to denote the firm's value function at stage 1 and stage 2, respectively. Firm l's problem at stage 1 is

$$\begin{aligned} v_1^f(\tilde{z}_l, \Sigma_l; \xi_1, X) &= \max_{H_l, K_l} E^*[v_2^f(\hat{\tilde{z}}_l', \Sigma_l'; \hat{\xi}_2, X)] \\ &\text{s.t.} \quad \text{Updating rules (7) and (8)} \end{aligned} \tag{3}$$

and firm l's value at stage 2 is

$$\begin{aligned} v_2^f(\tilde{z}_l', \Sigma_l'; \xi_2, X) &= \lambda(Y^{\frac{1}{\theta}} Y_l^{1-\frac{1}{\theta}} - WH_l - r^K K_l) + \beta \int v_1^f(\tilde{z}_l', \Sigma_l'; \xi_1', X') dF(X'|X) \\ \text{s.t.} \quad \xi_1' &= \Gamma(\xi_2, X) \end{aligned} \tag{4}$$

where we simplify the exposition by expressing a firm's value in terms of the marginal utility λ of the representative household. Similar to the household's problem, a firm's problem at stage 1 is to choose the labor and capital demand so as to maximize the worst-case stage 2 value. Note that the posterior mean $\tilde{z}_l'$ will be determined by the realization of output Y_l at stage 2 while the posterior variance Σ_l' is determined by Σ_l and the input level at stage 1.

The recursive competitive equilibrium is therefore a collection of value functions, policy functions, and prices such that

1. Households and firms optimize; (1) – (4).
2. The labor market, goods market, and asset markets clear.
3. The law of motion $\xi_1' = \Gamma(\xi_2, X)$ is induced by the firms' policy functions.

B Solution procedure

Here we describe the general solution procedure of the model. First, we derive the law of motion assuming that the model is a rational expectations model where the worst case expectations are on average correct. Second, we take the equilibrium law of motion formed under ambiguity and then evaluate the dynamics under the econometrician's data generating process. We provide a step-by-step description of the procedure:

1. Find the worst-case steady state.

   We first compute the steady state of the filtering problem (7), (8), and (11), under the worst-case mean to find the firm-level profitability at the worst-case steady state, $\bar{z}^0$. We then solve the steady state for other equilibrium conditions evaluated at $\bar{z}^0$.

2. Log-linearize the model around the worst-case steady state.

   We can solve for the dynamics using standard tools for linear rational expectation models. We base our discussion based on the method proposed by Sims (2002).

   We first need to deal with the issue that idiosyncratic shocks realize at the beginning of stage 2. Handling this issue correctly is important, since variables chosen at stage 1, such as input choice, should be based on the worst-case profitability, while variables chosen at stage 2, such as consumption and investment, would be based on the realized profitability (but also on the worst-case future profitability). To do this, we exploit the certainty equivalence property of linear decision rules. We first solve for decision rules *as if* both aggregate and idiosyncratic shocks realize at the beginning of the period. We call them "pre-production decision rules". We then solve for decision rules *as if* (i) both aggregate and idiosyncratic shocks realize at the beginning of the period and (ii) stage 1 variables are pre-determined. We call them "post-production decision rules". Finally, when we characterize the dynamics from the perspective of the econometrician, we combine the pre-production and post-production decision rules and obtain and equilibrium law of motion.

   To obtain pre-production decision rules, we collect the linearized equilibrium conditions, which include firm-level conditions, into the canonical form:

$$\mathbf{\Gamma}_0^{pre}\hat{\mathbf{y}}_t^{pre,0} = \mathbf{\Gamma}_1^{pre}\hat{\mathbf{y}}_{t-1}^{pre,0} + \mathbf{\Psi}^{pre}\varrho_t + \mathbf{\Upsilon}^{pre}\iota_t^{pre},$$

   where $\hat{\mathbf{y}}_t^{pre,0}$ is a column vector of size k that contains all variables and the conditional expectations. $\hat{\mathbf{y}}_t^{pre,0} = \mathbf{y}_t^{pre} - \bar{\mathbf{y}}^0$ denotes deviations from the worst-case steady state and ι_t are expectation errors, which we define as $\iota_t^{pre} = \hat{\mathbf{y}}_t^{pre,0} - E_{t-1}^*\hat{\mathbf{y}}_t^{pre,0}$ such that $E_{t-1}^*\iota_t^{pre} = 0$. We define $\varrho_t = [e_{l,t} \quad e_t]'$, where $e_{l,t} = [\epsilon_{z,l,t} \quad u_{l,t} \quad \nu_{l,t}]'$ is a vector of idiosyncratic shocks and e_t is a vector of aggregate shocks of size n.

   The vector $\hat{\mathbf{y}}_t^{pre,0}$ contains firm-level variables such as firm l's labor input, $H_{l,t}$. In contrast to other linear heterogeneous-agent models with imperfect information such as Lorenzoni (2009), all agents share the same information set. Thus, to derive the aggregate law of motion, we simply aggregate over firm l's linearized conditions and replace firm-specific variables with their cross-sectional means (e.g., we replace $H_{l,t}$ with $H_t \equiv \int_0^1 H_{l,t}dl$) and set $e_{l,t} = \mathbf{0}$, which uses the law of large numbers for idiosyncratic shocks.

   We order variables in $\hat{\mathbf{y}}_t^{pre,0}$ as

$$\hat{\mathbf{y}}_t^{pre,0} = \begin{bmatrix} \hat{\mathbf{y}}_{1,t}^{pre,0} \\ \hat{\mathbf{y}}_{2,t}^{pre,0} \\ \hat{\mathbf{s}}_t^{pre,0} \end{bmatrix},$$

   where $\hat{\mathbf{y}}_{1,t}^{pre,0}$ is a column vector of size k_1 of variables determined at stage 1, $\hat{\mathbf{y}}_{2,t}^{pre,0}$ is a column vector of size k_2 of variables determined at stage 2, and $\hat{\mathbf{s}}_t^{pre,0} = [\hat{s}_{1,t}^{pre,0} \quad \hat{s}_{2,t}^{pre,0}]'$, where $s_{1,t} = \bar{z} - E_{t-1}^* z_t$ and $s_{2,t} = \bar{z} - \tilde{z}_{t|t}$.

   The resulting solution of pre-production decision rules is obtained applying the method developed by Sims (2002):

$$\hat{\mathbf{y}}_t^{pre,0} = \mathbf{T}^{pre}\hat{\mathbf{y}}_{t-1}^{pre,0} + \mathbf{R}^{pre}[\mathbf{0}_{3\times 1} \quad e_t]', \tag{5}$$

where $\mathbf{T}^{pre}$ and $\mathbf{R}^{pre}$ are $k \times k$ and $k \times (n+3)$ matrices, respectively.

The solution of post-production decision rules can be obtained in a similar way by first collecting the equilibrium conditions into the canonical form

$$\mathbf{\Gamma}_0^{post} \hat{\mathbf{y}}_t^{post,0} = \mathbf{\Gamma}_1^{post} \hat{\mathbf{y}}_{t-1}^{post,0} + \mathbf{\Psi}^{post} \varrho_t + \mathbf{\Upsilon}^{post} \iota_t^{post},$$

and is given by

$$\hat{\mathbf{y}}_t^{post,0} = \mathbf{T}^{post} \hat{\mathbf{y}}_{t-1}^{post,0} + \mathbf{R}^{post} [\mathbf{0}_{3\times 1} \quad e_t]', \tag{6}$$

where

$$\hat{\mathbf{y}}_t^{post,0} = \begin{bmatrix} \hat{\mathbf{y}}_{1,t}^{post,0} \\ \hat{\mathbf{y}}_{2,t}^{post,0} \\ \hat{\mathbf{s}}_t^{post,0} \end{bmatrix},$$

and $\mathbf{T}^{post}$ and $\mathbf{R}^{post}$ are $k \times k$ and $k \times (n+3)$ matrices, respectively.

3. Characterize the dynamics from the econometrician's perspective.

   The above law of motion was based on the worst-case probabilities. We need to derive the equilibrium dynamics under the true DGP, where the cross-sectional mean of firm-level profitability is $\bar{z}$. We are interested in two objects: the zero-risk steady state and the dynamics around that zero-risk steady state.

   (a) Find the zero-risk steady state.

   This the fixed point $\bar{\mathbf{y}}$ where the decision rules (5) and (6) are evaluated at the realized cross-sectional mean of firm-level profitability $\bar{z}$:

$$\begin{aligned} \bar{\mathbf{y}}^{pre} - \bar{\mathbf{y}}^0 &= \mathbf{T}^{pre}(\bar{\mathbf{y}} - \bar{\mathbf{y}}^0), \\ \bar{\mathbf{y}}^{post} - \bar{\mathbf{y}}^0 &= \mathbf{T}^{post}(\bar{\mathbf{y}} - \bar{\mathbf{y}}^0) + \mathbf{R}^{post}[\bar{\mathbf{s}} \quad \mathbf{0}_{(n+1)\times 1}]', \end{aligned} \tag{7}$$

   where

$$\bar{\mathbf{y}} = \begin{bmatrix} \bar{\mathbf{y}}_1^{pre} \\ \bar{\mathbf{y}}_2^{post} \\ \bar{\mathbf{s}}^{post} \end{bmatrix}.$$

   Note that we do not feed in the realized firm-level profitability to the pre-production decision rules since idiosyncratic shocks realize at the beginning of stage 2.

   We obtain $\bar{\mathbf{s}}$ from

$$\bar{\mathbf{s}} = [\mathbf{T}_{3,1}^{post} \quad \mathbf{T}_{3,2}^{post} \quad \mathbf{T}_{3,3}^{post}](\bar{\mathbf{y}} - \bar{\mathbf{y}}^0) + \bar{\mathbf{s}}^0,$$

   where

$$\mathbf{T}^{post} = \begin{bmatrix} \underset{(k_1\times k_1)}{\mathbf{T}_{1,1}^{post}} & \underset{(k_1\times k_2)}{\mathbf{T}_{1,2}^{post}} & \underset{(k_1\times 2)}{\mathbf{T}_{1,3}^{post}} \\ \underset{(k_2\times k_1)}{\mathbf{T}_{2,1}^{post}} & \underset{(k_2\times k_2)}{\mathbf{T}_{2,2}^{post}} & \underset{(k_2\times 2)}{\mathbf{T}_{2,3}^{post}} \\ \underset{(2\times k_1)}{\mathbf{T}_{3,1}^{post}} & \underset{(2\times k_2)}{\mathbf{T}_{3,2}^{post}} & \underset{(2\times 2)}{\mathbf{T}_{3,3}^{post}} \end{bmatrix}.$$

(b) Dynamics around the zero-risk steady state.

Denoting $\hat{\mathbf{y}}_t \equiv \mathbf{y}_t - \bar{\mathbf{y}}$ the deviations from the zero-risk steady state, we combine the decision rules (5) and (6) evaluated at the true DGP and the equations for the zero-risk steady state (7):

$$\hat{\mathbf{y}}_t^{pre} = \mathbf{T}^{pre}\hat{\mathbf{y}}_{t-1} + \mathbf{R}^{pre}[\mathbf{0}_{3\times 1} \quad e_t]', \tag{8}$$

$$\hat{\mathbf{y}}_t^{post} = \mathbf{T}^{post}[\hat{\mathbf{y}}_{1,t}^{pre} \quad \hat{\mathbf{y}}_{2,t-1} \quad \hat{\mathbf{s}}_{t-1}]' + \mathbf{R}^{post}[\hat{\hat{\mathbf{s}}}_t \quad 0 \quad e_t]', \tag{9}$$

$$\hat{\hat{\mathbf{s}}}_t = [\mathbf{T}_{3,1}^{post} \quad \mathbf{T}_{3,2}^{post} \quad \mathbf{T}_{3,3}^{post}][\hat{\mathbf{y}}_{1,t}^{pre} \quad \hat{\mathbf{y}}_{2,t-1} \quad \hat{\mathbf{s}}_{t-1}]' + \mathbf{R}_{3,3}^{post}[\mathbf{0}_{3\times 1} \quad e_t]', \tag{10}$$

and

$$\hat{\mathbf{y}}_t = \begin{bmatrix} \hat{\mathbf{y}}_{1,t}^{pre} \\ \hat{\mathbf{y}}_{2,t}^{post} \\ \hat{\mathbf{s}}_t^{post} \end{bmatrix}, \tag{11}$$

where

$$\mathbf{R}^{post} = \begin{bmatrix} \underset{(k_1\times 2)}{\mathbf{R}_{1,1}^{post}} & \underset{(k_1\times 1)}{\mathbf{R}_{1,2}^{post}} & \underset{(k_1\times n)}{\mathbf{R}_{1,3}^{post}} \\ \underset{(k_2\times 2)}{\mathbf{R}_{2,1}^{post}} & \underset{(k_2\times 1)}{\mathbf{R}_{2,2}^{post}} & \underset{(k_2\times n)}{\mathbf{R}_{2,3}^{post}} \\ \underset{(2\times 2)}{\mathbf{R}_{3,1}^{post}} & \underset{(2\times 1)}{\mathbf{R}_{3,2}^{post}} & \underset{(2\times n)}{\mathbf{R}_{3,3}^{post}} \end{bmatrix}.$$

We combine equations (8), (9), (10), and (11) to obtain the equilibrium law of motion. To do so, we first define submatrices of $\mathbf{T}^{pre}$ and $\mathbf{R}^{pre}$:

$$\mathbf{T}^{pre} = \begin{bmatrix} \underset{(k_1\times k)}{\mathbf{T}_1^{pre}} \\ \underset{(k_2\times k)}{\mathbf{T}_2^{pre}} \\ \underset{(2\times k)}{\mathbf{T}_3^{pre}} \end{bmatrix}, \quad \mathbf{R}^{pre} = \begin{bmatrix} \underset{(k_1\times 3)}{\mathbf{R}_{1,1}^{pre}} & \underset{(k_1\times n)}{\mathbf{R}_{1,2}^{pre}} \\ \underset{(k_2\times 3)}{\mathbf{R}_{2,1}^{pre}} & \underset{(k_2\times n)}{\mathbf{R}_{2,2}^{pre}} \\ \underset{(2\times 3)}{\mathbf{R}_{3,1}^{pre}} & \underset{(2\times n)}{\mathbf{R}_{3,2}^{pre}} \end{bmatrix}.$$

A $k \times k$ matrix $\mathbf{T}$ is then given by

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1^{pre} \\ \mathbf{T}_2 \\ \mathbf{T}_3 \end{bmatrix},$$

where $\mathbf{T}_2$ and $\mathbf{T}_3$ are given by

$$\mathbf{T}_2 = [\mathbf{Q}_{2,1} \quad \mathbf{Q}_{2,2} + \mathbf{T}_{2,2}^{post} + \mathbf{R}_{2,1}^{post}\mathbf{T}_{3,2}^{post} \quad \mathbf{Q}_{2,3} + \mathbf{T}_{2,3}^{post} + \mathbf{R}_{2,1}^{post}\mathbf{T}_{3,3}^{post}],$$
$$\mathbf{T}_3 = [\mathbf{Q}_{3,1} \quad \mathbf{Q}_{3,2} + \mathbf{T}_{3,2}^{post} + \mathbf{R}_{3,1}^{post}\mathbf{T}_{3,2}^{post} \quad \mathbf{Q}_{3,3} + \mathbf{T}_{3,3}^{post} + \mathbf{R}_{3,1}^{post}\mathbf{T}_{3,3}^{post}],$$

and $\mathbf{Q}_{2,1}$, $\mathbf{Q}_{2,2}$, and $\mathbf{Q}_{2,3}$ are $k_2 \times k_1$, $k_2 \times k_2$, and $k_2 \times 2$ submatrices of $\mathbf{Q}_2$, where $\mathbf{Q}_2 \equiv (\mathbf{T}_{2,1}^{post} + \mathbf{R}_{2,1}^{post}\mathbf{T}_{3,1}^{post})\mathbf{T}_1^{pre}$, so that $\mathbf{Q}_2 = [\mathbf{Q}_{2,1} \quad \mathbf{Q}_{2,2} \quad \mathbf{Q}_{2,3}]$. Similarly, $\mathbf{Q}_{3,1}$, $\mathbf{Q}_{3,2}$, and $\mathbf{Q}_{3,3}$ are $k_3 \times k_1$, $k_3 \times k_2$, and $k_3 \times 2$ submatrices of $\mathbf{Q}_3$, where $\mathbf{Q}_3 \equiv (\mathbf{T}_{3,1}^{post} + \mathbf{R}_{3,1}^{post}\mathbf{T}_{3,1}^{post})\mathbf{T}_1^{pre}$, so that $\mathbf{Q}_3 = [\mathbf{Q}_{3,1} \quad \mathbf{Q}_{3,2} \quad \mathbf{Q}_{3,3}]$.

A $k \times n$ matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{1,2}^{pre} \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix},$$

where

$$\mathbf{R}_2 = \mathbf{T}_{2,1}^{post}\mathbf{R}_{1,2}^{pre} + \mathbf{R}_{2,1}^{post}(\mathbf{T}_{3,1}^{post}\mathbf{R}_{1,2}^{pre} + \mathbf{R}_{3,3}^{post}) + \mathbf{R}_{2,3}^{post},$$
$$\mathbf{R}_3 = \mathbf{T}_{3,1}^{post}\mathbf{R}_{1,2}^{pre} + \mathbf{R}_{3,1}^{post}(\mathbf{T}_{3,1}^{post}\mathbf{R}_{1,2}^{pre} + \mathbf{R}_{3,3}^{post}) + \mathbf{R}_{3,3}^{post}.$$

The equilibrium law of motion is then given by

$$\hat{\mathbf{y}}_t = \mathbf{T}\hat{\mathbf{y}}_{t-1} + \mathbf{R}e_t.$$

C Illustration of log-linearization and effects of idiosyncratic uncertainty

In what follows we explain the log-linearizing logic by simple expressions for the expected worst-case output at stage 1 (pre-production) and the realized output at stage 2 (post-production). We use the example to illustrate that uncertainty about the firm-level productivity has a first-order effect at the aggregate level. To do so, we first log-linearize the expected worst-case output of firm l at stage 1, as described in section Appendix B

$$E_t^*\hat{Y}_{l,t}^0 = \hat{A}_t^0 + E_t^*\hat{z}_{l,t}^0 + \hat{F}_{l,t}^0, \tag{12}$$

and the realized output of individual firm l at stage 2:

$$\hat{Y}_{l,t}^0 = \hat{A}_t^0 + \hat{z}_{l,t}^0 + \hat{F}_{l,t}^0, \tag{13}$$

where we use $\hat{x}_t^0 = x_t - \bar{x}^0$ to denote log-deviations from the worst-case steady state and set the trend growth rate γ to zero to ease notation. The worst-case individual output (12) is the sum of three components: the current level of economy-wide TFP, the worst-case individual TFP, and the input level. The realized individual output (13), in turn, is the sum of economy-wide TFP, the *realized* individual TFP, and the input level.

We then aggregate the log-linearized individual conditions (12) and (13) to obtain the cross-sectional mean of worst-case individual output:

$$E_t^*\hat{Y}_t^0 = \hat{A}_t^0 + E_t^*\hat{z}_t^0 + \hat{F}_t^0, \tag{14}$$

and the cross-sectional mean of realized individual output:

$$\hat{Y}_t^0 = \hat{A}_t^0 + \hat{z}_t^0 + \hat{F}_t^0, \tag{15}$$

where we simply eliminate subscript l to denote the cross-sectional mean, i.e., $\hat{x}_t^0 \equiv \int_0^1 \hat{x}_{l,t}^0 dl$.

We now characterize the dynamics under the true DGP. To do this, we feed in the cross-sectional mean of individual TFP, which is constant under the true DGP, into (14) and (15). Using (14), the cross-sectional mean of worst-case output is given by

$$E_t^*\hat{Y}_t = \hat{A}_t + E_t^*\hat{z}_t + \hat{F}_t, \tag{16}$$

where we use $\hat{x}_t = x_t - \bar{x}$ to denote log-deviations from the steady-state under the true DGP. Using (15), the realized aggregate output is given by

$$\hat{Y}_t = \hat{A}_t + \hat{F}_t, \tag{17}$$

where we used $\hat{z}_t = 0$ under the true DGP. Importantly, $E_t^*\hat{z}_t$ in (17) is not necessarily zero outside the steady state. To see this, combine (11) and (15) and log-linearize to obtain an expression for $E_t^*\hat{z}_{l,t}$:

$$E_t^*\hat{z}_{l,t} = \varepsilon_{z,z}\hat{\hat{z}}_{l,t-1|t-1} - \varepsilon_{z,\Sigma}\hat{\Sigma}_{l,t-1|t-1}. \tag{18}$$

From (8), the posterior variance is negatively related to the level of input F:

$$\hat{\Sigma}_{l,t-1|t-1} = \varepsilon_{\Sigma,\Sigma}\hat{\Sigma}_{l,t-2|t-2} - \varepsilon_{\Sigma,Y}\hat{F}_{l,t-1}, \tag{19}$$

The elasticities $\varepsilon_{z,z}$, $\varepsilon_{z,\Sigma}$, $\varepsilon_{\Sigma,\Sigma}$, and $\varepsilon_{\Sigma,Y}$ are functions of structural parameters and are all positive. We combine (18) and (19) to obtain

$$E_t^*\hat{z}_{l,t} = \varepsilon_{z,z}\hat{\hat{z}}_{l,t-1|t-1} - \varepsilon_{z,\Sigma}\varepsilon_{\Sigma,\Sigma}\hat{\Sigma}_{l,t-2|t-2} + \varepsilon_{z,\Sigma}\varepsilon_{\Sigma,Y}\hat{F}_{l,t-1}. \tag{20}$$

Finally, we aggregate (20) across all firms:

$$E_t^*\hat{z}_t = -\varepsilon_{z,\Sigma}\varepsilon_{\Sigma,\Sigma}\hat{\Sigma}_{t-2|t-2} + \varepsilon_{z,\Sigma}\varepsilon_{\Sigma,Y}\hat{F}_{t-1}, \tag{21}$$

where we used $\int_0^1 \hat{\hat{z}}_{l,t-1|t-1}dl = 0$.[2]

Notice again that the worst-case conditional cross-sectional mean simply aggregates linearly the worst-case conditional mean, $-a_{l,t}$, of each firm. Since the firm-specific worst-case means are a function of idiosyncratic uncertainty, which in turn depend on the firms' scale, equation (21) shows that the average level of economic activity, $\hat{F}_{t-1}$, has a first-order effect on the cross-sectional average of the worst-case mean.

D Quantitative model

D.1 Financial accelerator and financial shocks

We embed a Bernanke et al. (1999)-type financial accelerator mechanism by introducing an entrepreneurial sector that buys capital from households at price q_t at the end of period t and receives the proceed from production at the end of $t+1$ and resell it to households at price q_{t+1}. Entrepreneurs are ex-ante homogeneous and risk-neutral. They hold net worth N_t which could be used to partially finance their capital expenditures q_tK_t. Entrepreneurs face an exogenous survival rate ζ; when they exit the market, their net worth is rebated back to the households as a lump-sum transfer. The new entrepreneurs, who replace the entrepreneurs that exit the market, receive a start-up fund T_t^E which is financed via a lump-sum tax on households. Risk-neutral financial intermediaries provide external finance to entrepreneurs using funds obtained from households.

[2]This follows from aggregating the log-linearized version of (7) and evaluating the equation under the true DGP. Intuitively, since the cross-sectional mean of idiosyncratic TFP is constant, the cross-sectional mean of the Kalman posterior mean estimate is a constant as well.

After the realization of period $t+1$ aggregate shocks, entrepreneurs sign a debt contract with the financial intermediaries. Entrepreneurs then transform capital K_t purchased from households into effective units $\omega_{t+1}K_t$ that can be rented out to firms, where ω_{t+1} is an idiosyncratic shock that is unobservable to the financial intermediaries unless they pay a monitoring cost. We assume that ω is log-normally distributed with mean one: $\ln\omega \sim N(-0.5\sigma_\omega^2, \sigma_\omega^2)$. The loan contract is characterized by the level of capital q_tK_t and their associated level of borrowing $B_t = q_tK_t - N_t$, the loan rate Z_{t+1} and a cutoff value $\overline{\omega}_{t+1}$ for the idiosyncratic shock. The indifference condition for the entrepreneurs is given by

$$\overline{\omega}_{t+1}E^*_{t+1}R^K_{t+1}q_tK_t = Z_{t+1}B_t, \tag{22}$$

where R^K_{t+1} is evaluated under the worst-case expectation E^*_{t+1} since the contract is signed before the resolution of firm-level uncertainty. When $\omega_{t+1} > \overline{\omega}_{t+1}$, entrepreneurs repay the debt to the financial intermediaries and keep the difference $\omega_{t+1}R^K_{t+1}q_tK_t - Z_{t+1}B_t$. When $\omega_{t+1} \leq \overline{\omega}_{t+1}$, entrepreneurs declare bankruptcy and repay nothing while financial intermediaries pay a monitoring cost and recover the rest $(1-\mu)R^K_{t+1}q_tK_t$. The credit spread is defined as the difference between the loan rate and the risk-free rate: $Spread_t \equiv Z_{t+1} - R_t$.

The entrepreneur's problem is to choose (Z_{t+1}, B_t), to maximize their payoff

$$[1-\Gamma(\overline{\omega}_{t+1})]E^*_{t+1}R^K_{t+1}q_tK_t,$$

subject to the financial intermediaries' participation constraint (zero-profit condition), where $\Gamma(\overline{\omega}_{t+1}) \equiv \int_0^{\overline{\omega}_{t+1}} \omega f(\omega)d\omega + \overline{\omega}_{t+1}\int_{\overline{\omega}_{t+1}}^{\infty} f(\omega)d\omega$ and $f(\cdot)$ is the log-normal density from which ω is drawn. The solution to the problem is characterized by the first-order condition

$$E^*_t\left\{[1-\Gamma(\overline{\omega}_{t+1})]\frac{R^K_{t+1}}{R_t} + \frac{\Gamma'(\overline{\omega}_{t+1})}{\Gamma'(\overline{\omega}_{t+1}) - \mu G'(\overline{\omega}_{t+1})}\left(\frac{R^K_{t+1}}{R_t}[\Gamma(\overline{\omega}_{t+1}) - \mu G(\overline{\omega}_{t+1})] - \Delta^K_t - 1\right)\right\} = 0$$

, where $G(\overline{\omega}_{t+1}) \equiv \int_0^{\overline{\omega}_{t+1}} \omega f(\omega)d\omega$ and the zero-profit condition:

$$[\Gamma(\overline{\omega}_{t+1}) - \mu G(\overline{\omega}_{t+1})]E^*_{t+1}R^K_{t+1}q_tK_t - \Delta^K_t R_tB_t = R_tB_t, \tag{23}$$

where Δ^K_t is a financial shock that drives a wedge between the financial intermediaries' revenue (left-hand side) and its opportunity cost of its funds (right-hand side). Finally, the evolution of net worth is given by

$$N_{t+1} = \zeta(1-\Gamma(\widetilde{\omega}_{t+1}))R^K_{t+1}q_tK_t + (1-\zeta)T^E_t,$$

where $\widetilde{\omega}_{t+1}$ is the realized cutoff value, obtained by evaluating (22) under the realized return on capital.

D.2 Equilibrium conditions

As we describe above in Appendix B, we express equilibrium conditions from the perspective of agents at both stage 1 and stage 2. At stage 1, we need not only equilibrium conditions for variable determined before production (such as utilization and hours), but also those for variables determined after production (such as consumption and investment). At stage 2, we treat variables determined before production as pre-determined. To do this, we index period t variables determined at stage 1 by $t-1$ and period t variables determined at stage 2 by t. We then combine stage 1 and stage 2 equilibrium conditions by using the certainty equivalence property of linearized decision rules.

We use Ω_t^p to denote a "partial" information set at stage 1 in period t. This includes all predetermined variables and aggergate shocks at period t except for the period t monetary policy shock. Similarly, we use $\Omega_t^{p\prime}$ to denote a "partial" information set at stage 2 in period t. This includes all predetermined variables and aggergate and idiosyncratic shocks at period t except for the period t monetary policy shock.

We scale the variables in order to introduce stationary:

$$c_t = \frac{C_t}{\gamma^t}, y_{l,t} = \frac{Y_{l,t}}{\gamma^t}, k_{l,t-1} = \frac{K_{l,t-1}}{\gamma^t}, i_t = \frac{I_t}{\gamma^t}, w_t = \frac{W_t}{\gamma^t}, n_{t-1} = \frac{N_{t-1}}{\gamma^t}, t_t^E = \frac{T_t^E}{\gamma^t}, \tilde{\lambda}_t = \gamma^t \lambda_t, \tilde{\mu}_t = \gamma^t \mu_t,$$

where μ_t is the Lagrangian multiplier on the capital accumulation equation. We first describe the stage 1 equilibrium conditions.

Firms

An individual firm l's problem is to choose $\{U_{l,t}, K_{l,t}, H_{l,t}\}$ to maximize

$$E^* \sum_{s=0}^{\infty} \beta^{t+s} \lambda_{t+s} [P_{t+s}^W Y_{t+s}^{\frac{1}{\theta}} Y_{l,t+s}^{1-\frac{1}{\theta}} - W_{t+s} H_{l,t+s} - r_{t+s}^K K_{l,t+s-1} - a(U_{l,t+s}) K_{l,t+s-1} | \Omega_t^p],$$

where P_t^W is the price of whole-sale goods produced by firms and λ_t, and its detrended counterpart $\tilde{\lambda}_t$, is the marginal utility of the representative household:

$$\tilde{\lambda}_t = \frac{\gamma}{c_t - bc_{t-1}} - \beta b E^* \left[\frac{1}{\gamma c_{t+1} - bc_t} | \Omega_t^p \right], \tag{24}$$

subject to the following two constraints. The first constraint is the production function:

$$y_{l,t} = E^*[e^{A_t + z_{l,t}} f_{l,t} \overline{\nu}_{l,t} | \Omega_t^p], \tag{25}$$

where $\overline{\nu}_{l,t} \equiv \sum_{j=1}^{J_{l,t}} e^{\nu_{l,j,t}} / \overline{N}$ and $f_{l,t}$ is the input,

$$f_{l,t} = (U_{l,t} k_{l,t-1})^{\alpha} H_{l,t}^{1-\alpha}. \tag{26}$$

The worst case TFP $E_t^* z_{l,t+1|t+1}$ is given by

$$E_t^* z_{l,t+1} = \rho_z \tilde{z}_{l,t|t} - \eta \rho_z \sqrt{\Sigma_{l,t|t}}. \tag{27}$$

and the Kalman filter estimate $\tilde{z}_{l,t|t}$ evolves according to

$$\tilde{z}_{l,t|t} = \tilde{z}_{l,t|t-1} + \frac{\Sigma_{l,t|t-1}}{\Sigma_{l,t|t-1} + f_{l,t}^{-1} \sigma_{\nu}^2} \cdot (s_{l,t} - \tilde{z}_{l,t|t-1}). \tag{28}$$

The second constraint is the law of motion for posterior variance:

$$\Sigma_{l,t|t} = \left[\frac{\sigma_\nu^2}{f_{l,t}\Sigma_{l,t|t-1} + \sigma_\nu^2} \right] \Sigma_{l,t|t-1}. \tag{29}$$

As described in the main text, firms take into account the impact of their input choice on worst-case probabilities.

The first-order necessary conditions for firms' input choices are as follows:

- FONC for $\Sigma_{l,t|t}$

$$\begin{aligned} \psi_{l,t} =& \beta E^* \Bigg[\frac{1}{2} \tilde{\lambda}_{t+1} P_{t+1}^W \exp\left(A_{t+1} + \frac{\theta-1}{\theta} z_{l,t+1} \right) \left(\frac{\theta-1}{\theta} \right) \eta \rho_z \Sigma_{l,t|t}^{-\frac{1}{2}} f_{l,t+1} \\ &+ \psi_{l,t+1} \left\{ \frac{\sigma_\nu^2 \rho_z^2}{f_{l,t+1}(\rho_z^2 \Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2} - \frac{\sigma_\nu^2 \rho_z^2 (\rho_z^2 \Sigma_{l,t|t} + \sigma_z^2) f_{l,t+1}}{\{f_{l,t+1}(\rho_z^2 \Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2\}^2} \right\} |\Omega_t^p \Bigg], \end{aligned} \tag{30}$$

  where $\psi_{l,t}$ is the Lagrangian multiplier for the law of motion of posterior variance.

- FONC for $U_{l,t}$

$$\begin{aligned} &\tilde{\lambda}_t P_t^W \left(\frac{\theta-1}{\theta} \right) \alpha \frac{y_{l,t}}{U_{l,t}} + \psi_{l,t} \frac{\alpha \sigma_\nu^2 (\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2)^2 f_{l,t}}{\{f_{l,t}(\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2) + \sigma_\nu^2\}^2 U_{l,t}} \\ =& \tilde{\lambda}_t \{\chi_1 \chi_2 U_{l,t} + \chi_2 (1-\chi_1)\} k_{l,t-1} \end{aligned} \tag{31}$$

- FONC for $k_{l,t}$

$$r_t^K = P_t^W \left(\frac{\theta-1}{\theta} \right) \alpha \frac{y_{l,t}}{k_{l,t-1}} - a(U_{l,t}) + \frac{\psi_{l,t}}{\tilde{\lambda}_t} \cdot \frac{\alpha \sigma_\nu^2 (\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2)^2 f_{l,t}}{\{f_{l,t}(\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2) + \sigma_\nu^2\}^2 k_{l,t-1}} \tag{32}$$

- FONC for $H_{l,t}$

$$\tilde{\lambda}_t P_t^W \left(\frac{\theta-1}{\theta} \right) (1-\alpha) \frac{y_{l,t}}{H_{l,t}} + \psi_{l,t} \frac{(1-\alpha) \sigma_\nu^2 (\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2)^2 f_{l,t}}{\{f_{l,t}(\rho_z^2 \Sigma_{l,t-1|t-1} + \sigma_z^2) + \sigma_\nu^2\}^2 H_{l,t}} = \tilde{\lambda}_t \tilde{w}_t, \tag{33}$$

  where $\tilde{w}_t$ is the real wage: $\tilde{w}_t \equiv w_t / P_t$.

Firms sell their wholesale goods to monopolistically competitive retailers. Conditions associated with Calvo sticky prices are[3]

$$P_t^n = \tilde{\lambda}_t P_t^W y_t + \xi_p \beta E^* \left(\frac{\pi_{t+1}}{\bar{\pi}} |\Omega_t^p \right)^{\theta_p} P_{t+1}^n \tag{34}$$

$$P_t^d = \tilde{\lambda}_t y_t + \xi_p \beta E^* \left(\frac{\pi_{t+1}}{\bar{\pi}} |\Omega_t^p \right)^{\theta_p - 1} P_{t+1}^d \tag{35}$$

[3] We eliminate l-subscripts to denote cross-sectional means (e.g., $y_t \equiv \int_0^1 y_{l,t} dl$).

$$p_t^* = \left(\frac{\theta_p}{\theta_p - 1}\right)\frac{P_t^n}{P_t^d} \tag{36}$$

$$1 = (1-\xi_p)(p_t^*)^{1-\theta_p} + \xi_p\left(\frac{\bar{\pi}}{\pi_t}\right)^{1-\theta_p} \tag{37}$$

$$y_t^* = \tilde{p}_t^{-\theta_p} y_t \tag{38}$$

$$\tilde{p}_t = (1-\xi_p)(p_t^*)^{-\theta_p} + \xi_p\left(\frac{\bar{\pi}}{\pi_t}\right)^{-\theta_p} \tag{39}$$

Conditions associated with Calvo sticky wages are

$$v_t^1 = v_t^2 \tag{40}$$

$$v_t^1 = (w_t^*)^{1-\theta_w}\tilde{\lambda}_t H_t \tilde{w}_t + \xi_w \beta E^*\left(\frac{\pi_{t+1}^w w_{t+1}^*}{\bar{\pi} w_t^*}|\Omega_t^p\right)^{\theta_w - 1} v_{t+1}^1 \tag{41}$$

$$v_t^2 = \frac{\theta_w}{\theta_w - 1}(w_t^*)^{-\theta_w(1+\phi)} H_t^{1+\phi} + \xi_w \beta E^*\left(\frac{\pi_{t+1}^w w_{t+1}^*}{\bar{\pi} w_t^*}|\Omega_t^p\right)^{\theta_w(1+\phi)} v_{t+1}^2 \tag{42}$$

$$1 = (1-\xi_w)(w_t^*)^{1-\theta_w} + \xi_w E^*\left(\frac{\bar{\pi}}{\pi_t^w}|\Omega_t^p\right)^{1-\theta_w} \tag{43}$$

$$\pi_t^w = \pi_t \tilde{w}_t / \tilde{w}_{t-1} \tag{44}$$

Households

Households' Euler equation for risk-free bond:

$$\gamma\tilde{\lambda}_t = \beta E^*\left[\tilde{\lambda}_{t+1}\frac{R_t}{\pi_{t+1}}|\Omega_t^p\right] \tag{45}$$

Households' FONC for i_t

$$\begin{aligned}\gamma\tilde{\lambda}_t =& \gamma\tilde{\mu}_t\zeta_t\left[1 - \frac{\kappa}{2}\left(\frac{\gamma i_t}{i_{t-1}} - \gamma\right)^2 - \kappa\left(\frac{\gamma i_t}{i_{t-1}} - \gamma\right)\frac{\gamma i_t}{i_{t-1}}\right] \\ &+ \beta E^*\left[\tilde{\mu}_{t+1}\zeta_{t+1}\kappa\left(\frac{\gamma i_{t+1}}{i_t} - \gamma\right)\left(\frac{\gamma i_{t+1}}{i_t}\right)^2|\Omega_t^p\right]\end{aligned} \tag{46}$$

and the capital accumulation equation:

$$\gamma k_t = (1-\delta)k_{t-1} + \left\{1 - \frac{\kappa}{2}\left(\frac{\gamma i_t}{i_{t-1}} - \gamma\right)^2\right\}\zeta_t i_t. \tag{47}$$

Entrepreneurial sector

Entrepreneurs' optimality condition:

$$E^*\left\{[1-\Gamma(\overline{\omega}_{t+1})]\frac{R^K_{t+1}}{R_t}+\frac{\Gamma'(\overline{\omega}_{t+1})}{\Gamma'(\overline{\omega}_{t+1})-\mu G'(\overline{\omega}_{t+1})}\left(\frac{R^K_{t+1}}{R_t}[\Gamma(\overline{\omega}_{t+1})-\mu G(\overline{\omega}_{t+1})]-\Delta^K_t-1\right)|\Omega^p_t\right\}=0 \tag{48}$$

and the financial intermediaries' participation constraint:

$$[\Gamma(\overline{\omega}_t)-\mu G(\overline{\omega}_t)]R^K_t q_{t-1}k_{t-1}-\Delta^K_{t-1}R_{t-1}(q_{t-1}k_{t-1}-n_{t-1})=R_{t-1}(q_{t-1}k_{t-1}-n_{t-1}), \tag{49}$$

where the return on capital R^K_t is defined as

$$R^K_t=\{r^K_t+q_t(1-\delta)\}\times\frac{\pi_t}{q_{t-1}}, \tag{50}$$

and

$$q_t=\tilde{\mu}_t/\tilde{\lambda}_t. \tag{51}$$

The law of motion of net worth is given by

$$\gamma n_t=\zeta(1-\Gamma(\overline{\omega}_t))R^K_t q_{t-1}k_{t-1}+(1-\zeta)t^E_t, \tag{52}$$

where we assume that the transfer to the new entrepreneurs is constant: $t^E_t=t^E$.

We use the indifference condition by the entrepreneurs to pin down the loan rate Z_t:

$$\overline{\omega}_t R^K_t q_{t-1}k_{t-1}=Z_t(q_{t-1}k_{t-1}-n_{t-1}), \tag{53}$$

which we use to compute the credit spread: $Spread_t=Z_{t+1}-R_t$.

Monetary policy and resource constraint

Monetary policy rule:

$$\hat{R}_t=\rho_R\hat{R}_{t-1}+\sum_{i=0}^{2}\phi^i_\pi\hat{\pi}_{t-i}+\sum_{i=0}^{2}\phi^i_Y\Delta\hat{y}_{t-i}+\epsilon_{R,t} \tag{54}$$

Resource constraint:

$$c_t+i_t=(1-\bar{g})y_t, \tag{55}$$

where we have ignored the small terms arising from entrepreneurial default costs.

The 32 endogenous variables we solve are:

$$k_t, y_t, i_t, c_t, H_t, U_t, f_t, \tilde{\lambda}_t, \tilde{\mu}_t, \psi_t, r^K_t, R_t, R^K_t, q_t, E^*_t z_{t+1}, \tilde{z}_{t|t}, \Sigma_{t|t},$$
$$P^W_t, P^n_t, P^d_t, p^*_t, \pi_t, y^*_t, \tilde{p}_t, v^1_t, v^2_t, \tilde{w}_t, w^*_t, \pi^w_t, \overline{\omega}_t, n_t, Z_t$$

We have listed 32 conditions above, from (24) to (55). Of the above 32 endogenous variables, those that are determined at stage 1 are:

$$H_t, U_t, f_t, v^1_t, v^2_t, \tilde{w}_t, w^*_t, \pi^w_t, Z_t$$

We now describe the state 2 equilibrium conditions. To avoid repetitions, we only list conditions that are different from the state 1 conditions.

- (25):
$$y_{l,t} = e^{A_t+z_{l,t}} f_{l,t-1} \overline{\nu}_{l,t},$$
- (26):
$$f_{l,t} = (U_{l,t} k_{l,t})^{\alpha} H_{l,t}^{1-\alpha}$$
- (28):
$$\tilde{z}_{l,t|t} = \tilde{z}_{l,t|t-1} + \frac{\Sigma_{l,t|t-1}}{\Sigma_{l,t|t-1} + f_{l,t-1}^{-1}\sigma_\nu^2} \cdot (s_{l,t} - \tilde{z}_{l,t|t-1})$$
- (29):
$$\Sigma_{l,t|t} = \left[\frac{\sigma_\nu^2}{f_{l,t-1}\Sigma_{l,t|t-1} + \sigma_\nu^2}\right]\Sigma_{l,t|t-1}$$
- (30):
$$\begin{aligned}\psi_{l,t} =& \beta E^*\Bigg[\frac{1}{2}\tilde{\lambda}_{t+1} P_{t+1}^W \exp\left(A_{t+1} + \frac{\theta-1}{\theta} z_{l,t+1}\right)\left(\frac{\theta-1}{\theta}\right)\eta\rho_z \Sigma_{l,t|t}^{-\frac{1}{2}} f_{l,t} \\ &+ \psi_{l,t+1}\left\{\frac{\sigma_\nu^2\rho_z^2}{f_{l,t}(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2} - \frac{\sigma_\nu^2\rho_z^2(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) f_{l,t}}{\{f_{l,t}(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2\}^2}\right\}|\Omega_t^{p\prime}\Bigg]\end{aligned}$$
- (31):
$$\begin{aligned}&E^*\left[\tilde{\lambda}_{t+1} P_{t+1}^W \left(\frac{\theta-1}{\theta}\right)\alpha\frac{y_{l,t+1}}{U_{l,t}} + \psi_{l,t+1}\frac{\alpha\sigma_\nu^2(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2)^2 f_{l,t}}{\{f_{l,t}(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2\}^2 U_{l,t}}|\Omega_t^{p\prime}\right] \\ =& E^*[\tilde{\lambda}_{t+1}\{\chi_1\chi_2 U_{l,t} + \chi_2(1-\chi_1)\} k_{l,t}|\Omega_t^{p\prime}]\end{aligned}$$
- (32):
$$r_t^K = P_t^W\left(\frac{\theta-1}{\theta}\right)\alpha\frac{y_{l,t}}{k_{l,t-1}} - a(U_{l,t-1}) + \frac{\psi_{l,t}}{\tilde{\lambda}_t}\cdot\frac{\alpha\sigma_\nu^2(\rho_z^2\Sigma_{l,t-1|t-1} + \sigma_z^2)^2 f_{l,t-1}}{\{f_{l,t-1}(\rho_z^2\Sigma_{l,t-1|t-1} + \sigma_z^2) + \sigma_\nu^2\}^2 k_{l,t-1}}$$
- (33):
$$E^*\left[\tilde{\lambda}_{t+1} P_{t+1}^W\left(\frac{\theta-1}{\theta}\right)(1-\alpha)\frac{y_{l,t+1}}{H_{l,t}} + \psi_{l,t+1}\frac{(1-\alpha)\sigma_\nu^2(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) f_{l,t}}{\{f_{l,t}(\rho_z^2\Sigma_{l,t|t} + \sigma_z^2) + \sigma_\nu^2\}^2 H_{l,t}}|\Omega_t^{p\prime}\right] = E^*[\tilde{\lambda}_{t+1}\tilde{w}_t|\Omega_t^{p\prime}]$$
- (41):
$$v_t^1 = (w_t^*)^{1-\theta_w} E^*[\tilde{\lambda}_{t+1} H_t \tilde{w}_t|\Omega_t^{p\prime}] + \xi_w\beta E^*\left\{\left(\frac{\pi_{t+1}^w w_{t+1}^*}{\bar{\pi} w_t^*}\right)^{\theta_w-1} v_{t+1}^1|\Omega_t^{p\prime}\right\}$$
- (44):
$$\pi_t^w = E^*[\pi_{t+1}\tilde{w}_t/\tilde{w}_{t-1}|\Omega_t^{p\prime}]$$

D.3 Estimation method

We closely follow Christiano et al. (2010)'s description of the methodology. The Bayesian estimation of impulse-response matching first calculates the "likelihood" of the data using approximation based on standard asymptotic distribution theory. Let $\hat{\psi}$ denote the impulse response function computed from an identified SVAR and let $\psi(\theta)$ denote the impulse response function from the DSGE model, which depend on the structural parameters θ. Suppose the DSGE model as well as the SVAR specifications are correct and let θ_0 denote the true parameter vector; hence $\psi(\theta_0)$ is the true impulse response function. Then we have

$$\sqrt{T}(\hat{\psi} - \psi(\theta_0)) \xrightarrow{d} N(0, W(\theta_0)),$$

where T is the number of observations and $W(\theta_0)$ is the asymptotic sampling variance, which depends on θ_0. The asymptotic distribution of $\hat{\psi}$ can be rewritten as

$$\hat{\psi} \xrightarrow{d} N(\psi(\theta_0), V), \qquad V \equiv \frac{W(\theta_0)}{T}.$$

We use a consistent estimator of V, where the non-diagonal terms are set to zero and the main diagonal elements consist of the sample variance of $\hat{\psi}$.[4] As Christiano et al. (2011) describe in detail, this strategy improves small sample efficiency and can be justified in a way that is analogous to the estimation of frequency-zero spectral densities in Newey and West (1987). An additional advantage of this approach is that the interpretation of the estimator is graphically intuitive and transparent: it chooses parameters so that the model-implied impulse responses lie inside a confidence interval around the empirical responses. In contrast, when the non-diagonal terms of V are non-zero, the estimator also takes into account the deviations of the model from data across different impulse responses in a matter that is intractable.

The method then computes the likelihood

$$\mathcal{L}(\psi|\theta) = (2\pi)^{-\frac{N}{2}}|V|^{-\frac{1}{2}} \exp\{-0.5[\hat{\psi} - \psi(\theta)]'V^{-1}[\hat{\psi} - \psi(\theta)]\},$$

where N is the total number of elements in the impulse responses to be matched. Intuitively, the likelihood is higher when the model-based impulse response $\psi(\theta)$ is closer to the empirical counterpart $\hat{\psi}$, adjusting for the precision of the estimated empirical responses. We use the Bayes law to obtain the posterior distribution $p(\theta|\psi)$:

$$p(\theta|\psi) = \frac{p(\theta)\mathcal{L}(\psi|\theta)}{p(\psi)},$$

where $p(\theta)$ is the prior and $p(\psi)$ is the marginal likelihood. In each estimation, we generate 210,000 draws from the posterior distribution using the random-walk Metropolis-Hastings algorithm. We discard the first 10,000 draws as a burn-in period. In Table 1, we report the Brooks-Gelman-Rubin Potential Scale Reduction Factor (PSRF) by Gelman and Rubin (1992) and Brooks and Gelman (1998) for each parameter for our baseline model with ambiguity in the four shocks estimation. The PSRF values in the Table and all other estimations are well below the benchmark value of 1.1 considered as an upper bound for convergence.

[4] Christiano et al. (2005) and Altig et al. (2011) also use this approach in a frequentist context.

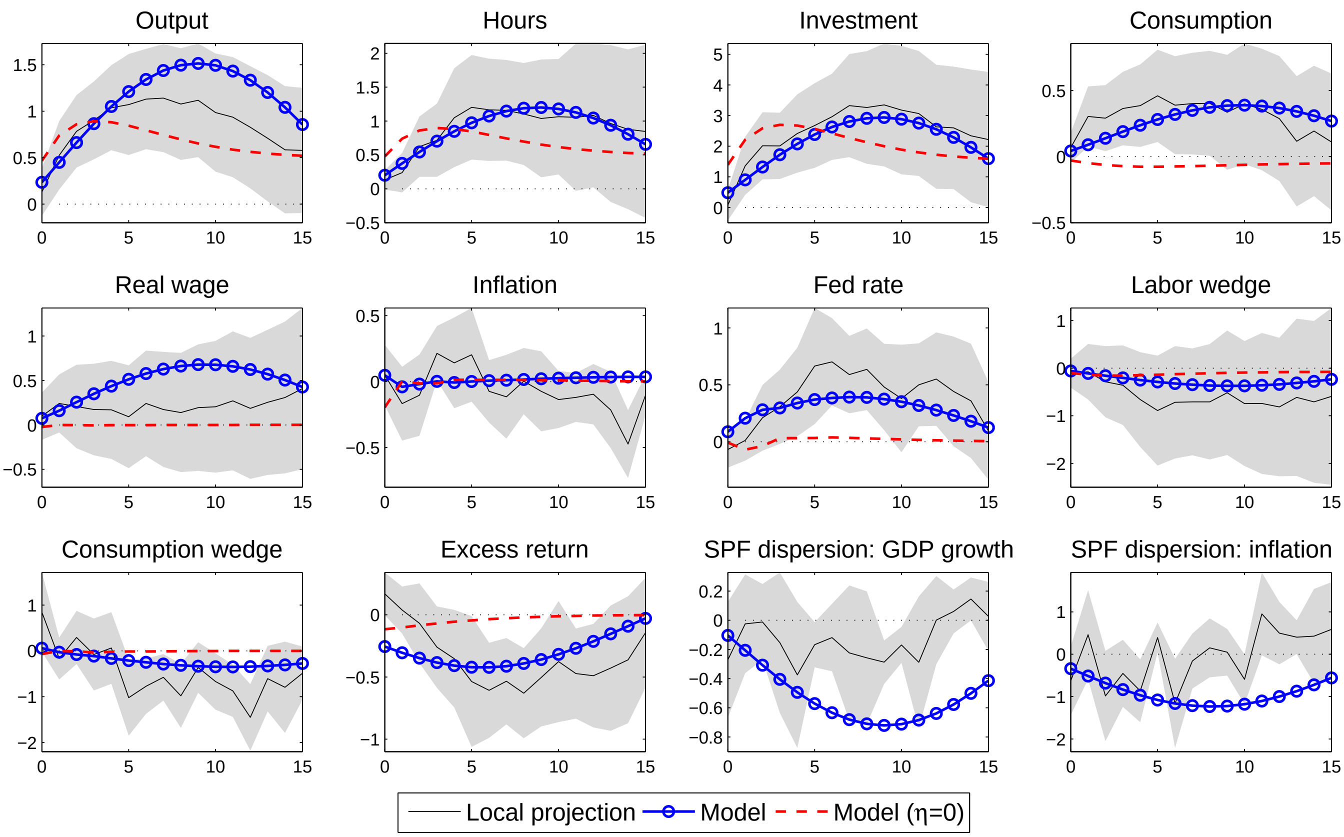
Table 1: Potential scale reduction factor

Parameter	PSRF	Parameter	PSRF	Parameter	PSRF	Parameter	PSRF
α	1.0038	ϕ	1.0010	χ_1	1.0010	b	1.0018
κ	1.0010	$\frac{1}{1-\xi_p}$	1.0001	$\frac{1}{1-\xi_w}$	1.0013	σ_ω	1.0004
μ	1.0018	Δ^K	1.0022	ρ_z	1.0045	σ_z	1.0002
0.5η	1.0008	$\bar{\Sigma}$	1.0013	ρ_R	1.0000	ϕ_π^0	1.0025
ϕ_π^1	1.0046	ϕ_π^2	1.0019	ϕ_Y^0	1.0046	ϕ_Y^1	1.0009
ϕ_Y^2	1.0001	ρ_Δ	1.0006	$100\sigma_\Delta$	1.0065	ρ_A	1.0037
$100\sigma_A$	1.0009	ρ_ζ	1.0036	$100\sigma_\zeta$	1.0070	$100\sigma_R$	1.0075

We use marginal likelihoods, computed from the MCMC output using the Geweke (1999)'s modified harmonic mean estimator, to perform model comparisons. Inoue and Shintani (2018) provide asymptotic justification for a such exercise. In particular, they show that as the sample size approaches infinity, a model with a higher marginal likelihood is either correct or a better approximation to true impulse responses.

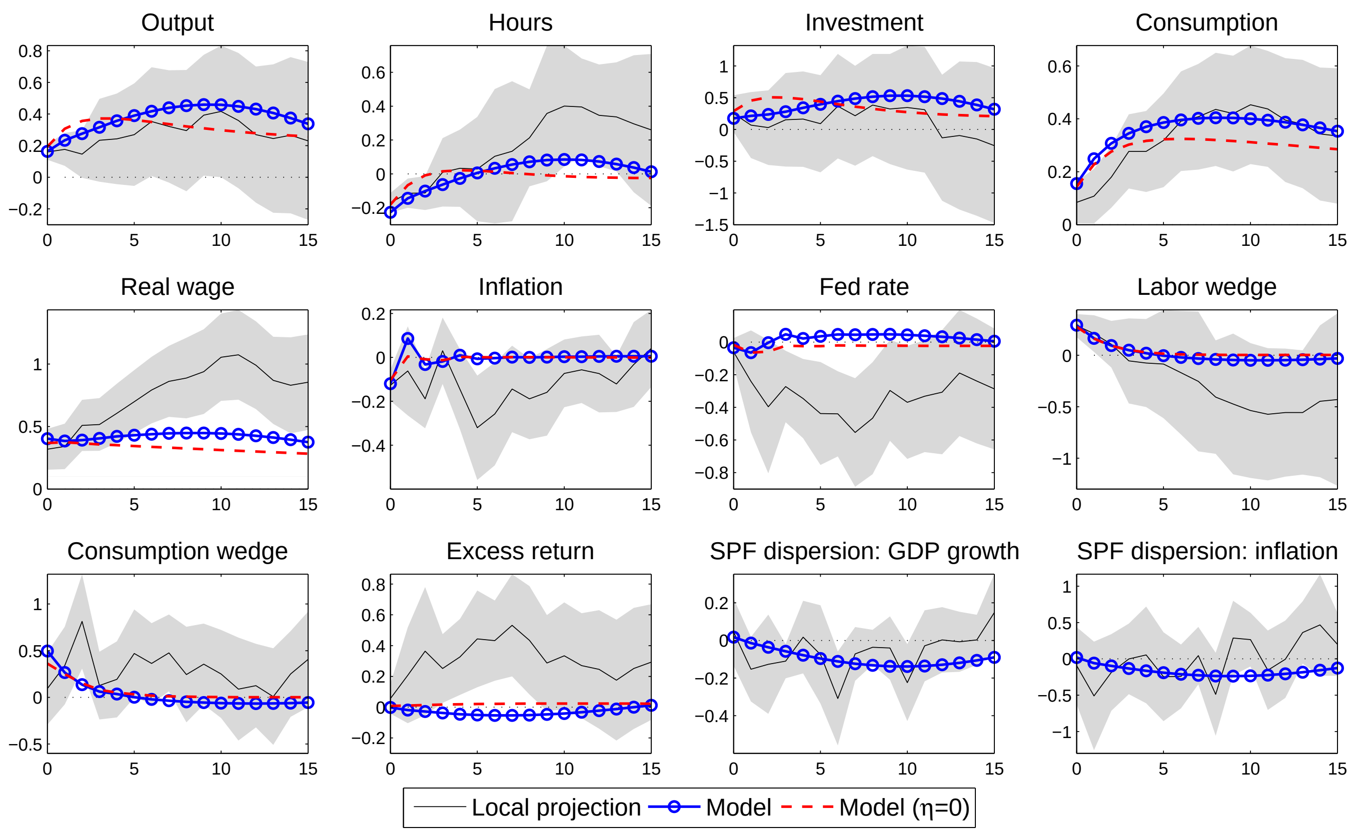
D.4 Additional figures

Figure 1: Responses to a financial shock



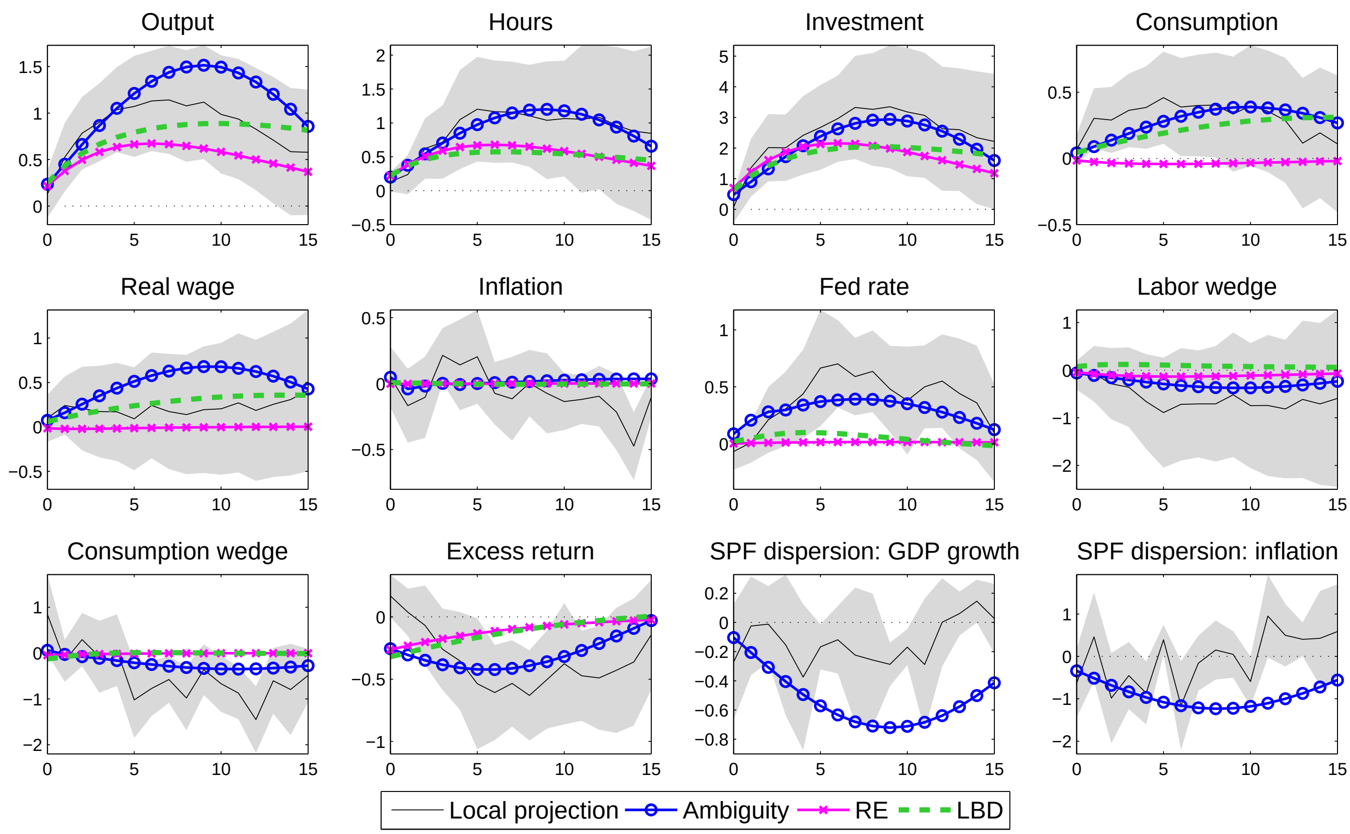
Notes: The black lines are the mean responses from the local projection and the shaded areas are the 95% confidence bands. The blue circled lines are the impulse responses from the baseline model with ambiguity, estimated using the local projection responses to all four structural shocks (technology, investment-specific, financial, and monetary policy). The red dashed lines are the counterfactual responses where we set the entropy constraint η to 0, while holding other parameters at the estimated values. The responses of output, hours, investment, consumption, and real wages are in percentage deviations from the steady states while inflation, fed rate, and excess return are in annual percentage points. The rest are in quarterly percentage points.

Figure 2: Responses to a neutral technology shock



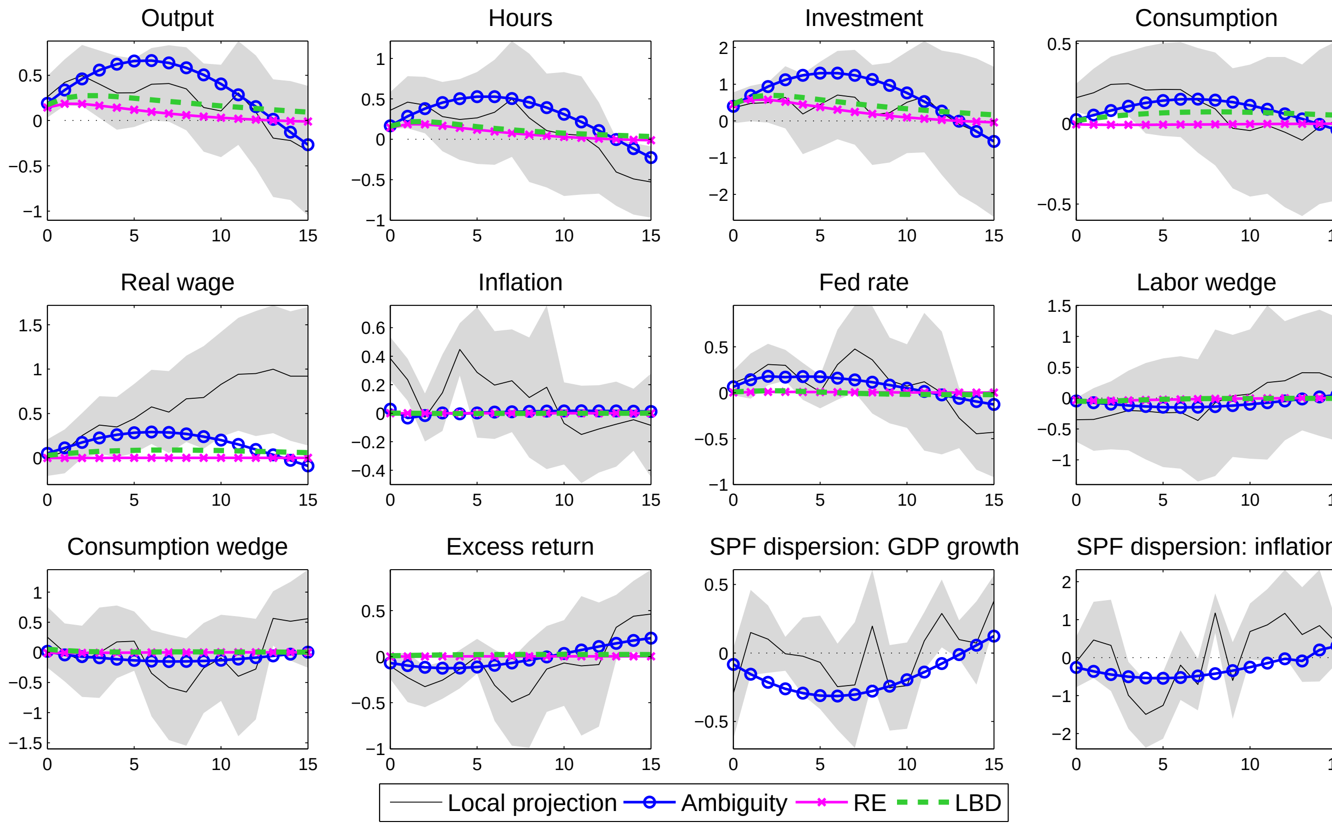
Notes: See notes from Figure 1.

Figure 3: Responses to a financial shock



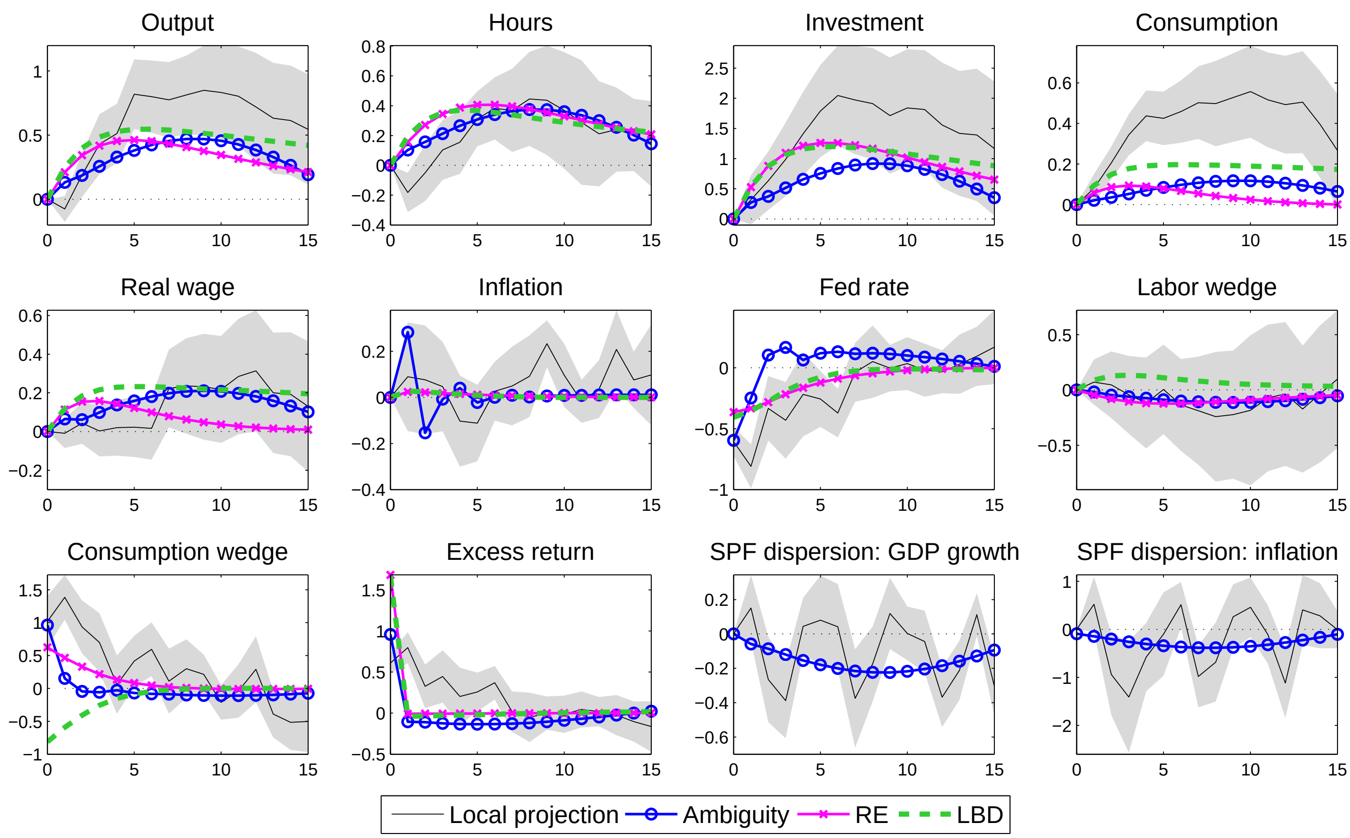
Notes: The black lines are the mean responses from the local projection and the shaded areas are the 95% confidence band. The blue circled lines are the impulse responses from the baseline model with ambiguity. The purple lines are the impulse responses from the standard RE model. The green dashed lines are the impulse responses from the LBD model. The impulse responses are estimated using the local projection responses to all four structural shocks (technology, investment-specific, financial, and monetary policy). The responses of output, hours, investment, consumption and real wages are in percentage deviations from the steady states while inflation, fed rate, and excess return are in annual percentage points. The rest are in quarterly percentage points.

Figure 4: Responses to an investment-specific technology shock



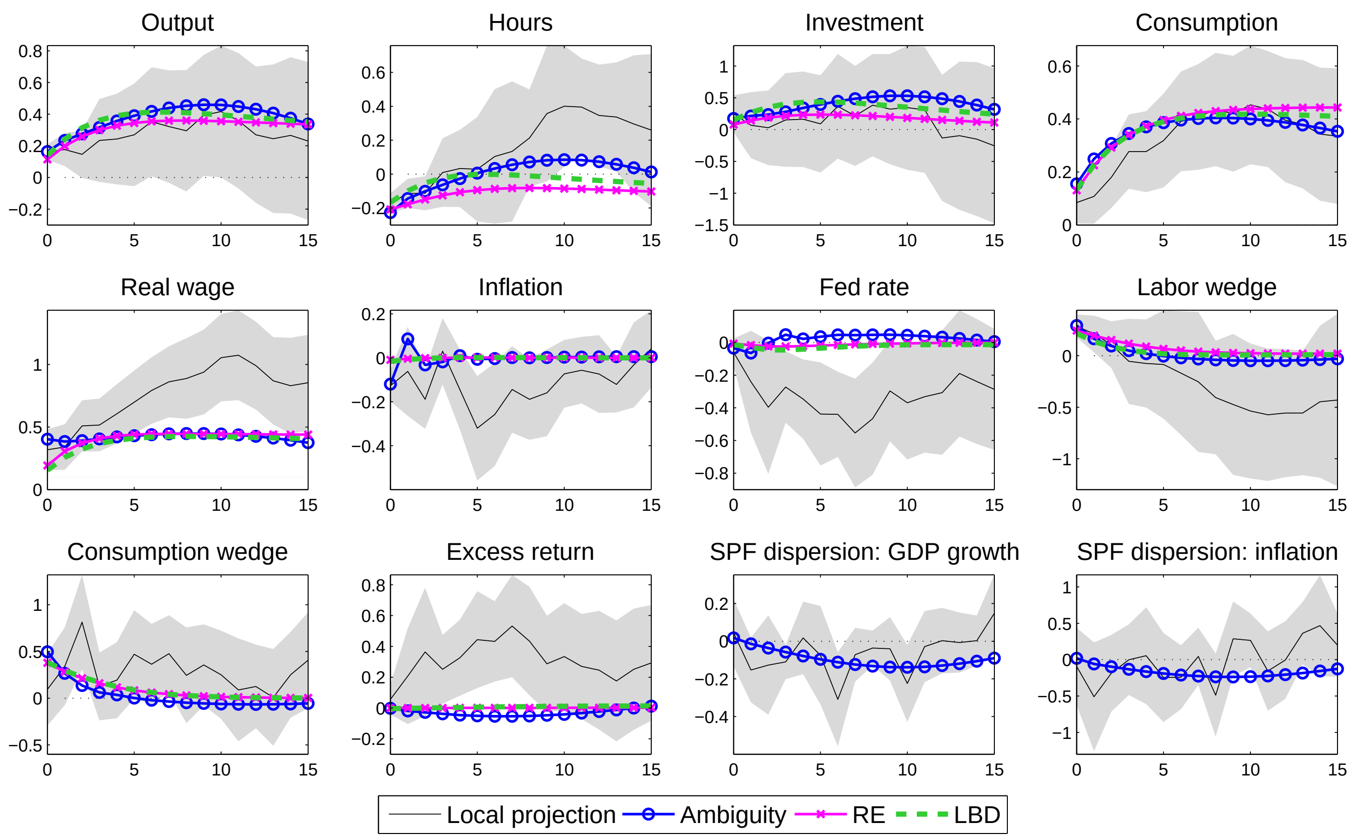
Notes: See notes from Figure 3.

Figure 5: Responses to a monetary policy shock



Notes: See notes from Figure 3.

Figure 6: Responses to a neutral technology shock



Notes: See notes from Figure 3.

E Data sources

We use the following data:

1. Real GDP in chained dollars, BEA, NIPA table 1.1.6, line 1.
2. GDP, BEA, NIPA table 1.1.5, line 1.
3. Personal consumption expenditures on nondurables, BEA, NIPA table 1.1.5, line 5.
4. Personal consumption expenditures on services, BEA, NIPA table 1.1.5, line 6.
5. Gross private domestic fixed investment (nonresidential and residential), BEA, NIPA table 1.1.5, line 8.
6. Personal consumption expenditures on durable goods, BEA, NIPA table 1.1.5, line 4.
7. Nonfarm business hours worked, BLS PRS85006033.
8. Nonfarm business hourly compensation, BLS PRS85006103.
9. Civilian noninstitutional population (16 years and over), BLS LNU00000000.
10. Effective federal funds rate, Board of Governors of the Federal Reserve System.
11. Capacity utilization index, Board of Governors of the Federal Reserve System.
12. Credit spread (GZ spread) constructed by Gilchrist and Zakrajšek (2012).
13. Return on assets of U.S. financial corporate sector constructed by Gilchrist and Zakrajšek (2012).

We then conduct the following transformations of the above data:

14. Real per capita GDP: (1)/(9)
15. GDP deflator: (2)/(1)
16. Real per capita consumption: [(3)+(4)]/[(9)×(15)]
17. Real per capita investment: [(5)+(6)]/[(9)×(15)]
18. Per capita hours: (7)/(9)
19. Real wages: (8)/(15)

References

Altig, D., L. J. Christiano, M. Eichenbaum, and J. Lindé (2011, April). Firm-specific capital, nominal rigidities and the business cycle. *Review of Economic Dynamics 14*(2), 225–247.

Bernanke, B. S., M. Gertler, and S. Gilchrist (1999). The financial accelerator in a quantitative business cycle framework. In J. B. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*, Volume 1, Part C, Chapter 21, pp. 1341–1393. Amsterdam: Elsevier.

Brooks, S. P. and A. Gelman (1998). General methods for monitoring convergence of iterative simulations. *Journal of computational and graphical statistics 7*(4), 434–455.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy 113*(1), 1–45.

Christiano, L. J., M. Trabandt, and K. Walentin (2010). Involuntary unemployment and the business cycle. NBER Working Paper.

Christiano, L. J., M. Trabandt, and K. Walentin (2011). Dsge models for monetary policy analysis. In B. M. Friedman and M. Woodford (Eds.), *Handbook of Monetary Economics*, Volume 3A, Chapter 7, pp. 285–367. Elsevier.

Gelman, A. and D. B. Rubin (1992). Inference from iterative simulation using multiple sequences. *Statistical science*, 457–472.

Geweke, J. (1999). Using simulation methods for bayesian econometric models: Inference, development, and communication. *Econometric Reviews 18*, 1–126.

Gilchrist, S. and E. Zakrajšek (2012). Credit spreads and business cycle fluctuations. *American Economic Review 102*(3), 1692–1720.

Inoue, A. and M. Shintani (2018). Quasi-bayesian model selection. *Quantitative Economics 9*, 1265–1297.

Lorenzoni, G. (2009). A theory of demand shocks. *American Economic Review 99*(5), 2050–2084.

Newey, W. K. and K. D. West (1987, May). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. *Econometrica 55*(3), 703–708.

Senga, T. (2018). A new look at uncertainty shocks: Imperfect information and misallocation. *Working Paper*.

Sims, C. A. (2002). Solving linear rational expectations models. *Computational Economics 20*(1), 1–20.